The Hurwitz enumeration problem of branched covers and Hodge integrals

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Abstract

We use algebraic methods to compute the simple Hurwitz numbers for arbitrary source and target Riemann surfaces. For an elliptic curve target, we reproduce the results previously obtained by string theorists. Motivated by the Gromov–Witten potentials, we find a general generating function for the simple Hurwitz numbers in terms of the representation theory of the symmetric group S_n. We also find a generating function for Hodge integrals on the moduli space $\overline{\mathcal{M}}_{g,2}$ of Riemann surfaces with two marked points, similar to that found by Faber and Pandharipande for the case of one marked point.
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1. Introduction

Many classical problems in enumerative geometry have been receiving renewed interests in recent years, the main reason being that they can be translated into the modern language of Gromov–Witten theory and, moreover, that they can be consequently solved. One such classical problem which has been under recent active investigation is the Hurwitz enumeration problem of counting topologically distinct, almost simple, ramified covers of the

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Nomenclature

$\mathcal{B}_{n,h,\sigma}$	see (2.2)
f^{γ}	the dimension of the irreducible representation $\gamma \in \mathcal{R}_n$
$H_{g,n}$	simplified notation for $\mu_{0,n}^{g,n}$. Not to be confused with H_h^g
H_h^g	generating functions for $\mu_{h,n}^{g,n}$, for fixed g and h (see (4.1))
$\tilde{H}_h^g$	generating functions for $\tilde{\mu}_{h,n}^{g,n}$, for fixed g and h
$\mathcal{R}_n$	the set of all ordinary irreducible representations of the symmetric group S_n
t_k^p	entries of the branching type matrix σ
$\hat{t}_j^i$	coordinates on the large phase space in the Gromov–Witten theory
$\mathcal{T}_{n,h,\sigma}$	subset of $\mathcal{B}_{n,h,\sigma}$, generating a transitive subgroup of S_n

Greek letters

$\mu_{h,n}^{g,n}$	the usual degree-n simple Hurwitz numbers for covers of a genus-h Riemann surface by genus-g Riemann surfaces
$\tilde{\mu}_{h,n}^{g,n}$, $N_{n,h,r}$	Mednykh's definition of simple Hurwitz numbers, including the fixed point contributions of the S_n action (see Section 2.2 for details)
$\chi_{\gamma}(1^{\alpha_1} \cdots n^{\alpha_n})$	the character of the irreducible representation $\gamma \in \mathcal{R}_n$ evaluated at the conjugacy class $[(1^{\alpha_1} \cdots n^{\alpha_n})]$. For those α_i which are zero, we omit the associated cycle in our notation

projective line or more generally of any compact connected Riemann surface. The almost simplicity condition is that the branch points be all simple with the possible exception of one degenerate point, often called ∞, the branching type of whose pre-images being specified by an ordered partition α of the degree n of the covering. Let $\alpha = (\alpha_1, \ldots, \alpha_w)$ be an ordered partition of n, denoted by $\alpha \vdash n$, of length $|\alpha| = w$. Then, the number r of simple branch points is determined by the Riemann–Hurwitz formula to be:

$$r = (1 - 2h)n + w + 2g - 2, \tag{1.1}$$

where h and g are the genera of the target and the source Riemann surfaces, respectively. The number $\mu_{h,w}^{g,n}(\alpha)$ of such covers is called the almost simple Hurwitz number, and in this paper, we mostly restrict ourselves to simple Hurwitz numbers $\mu_{h,n}^{g,n}(1^n)$, for which there is no ramification over ∞. Hurwitz numbers appear in many branches of mathematics and physics. In particular, they arise naturally in combinatorics, as they count factorizations of permutations into transpositions, and the original idea of Hurwitz expresses them in terms of the representation theory of the symmetric group. Indeed in this respect, the most general problem of counting covers of Riemann surfaces by Riemann surfaces, both reducible and irreducible, with arbitrary branch types, has been completely solved by Mednykh [14,15]. His formulas however generally do not allow explicit computations of the numbers, except in a few cases.

It turns out that one can successfully obtain the simple Hurwitz numbers using Mednykh's works, and in this paper, we shall compute them at low degrees for arbitrary target and source

Riemann surfaces. Hurwitz numbers also appear in physics: when the target is an elliptic curve, they are (up to overall normalization constants) the coefficients in the expansion of the free energies of the large N two-dimensional quantum Yang–Mills theory on the elliptic curve, which has in fact a string theory interpretation [11]. The total free energy and the partition function, which is its exponential, can be thought of as generating functions for simple Hurwitz numbers $\mu_{1,n}^{g,n}$. Generalizing this analogy, we have determined the generating functions for arbitrary targets in terms of the representation theory of the symmetric group S_n.

In the framework of Gromov–Witten theory, simple Hurwitz numbers can be considered as certain cohomological classes evaluated over the virtual fundamental class of the moduli space of stable maps to $\mathbb{P}^1$ [7]. By exploiting this reformulation, many new results such as new recursion relations [7,18] have been obtained. Furthermore, a beautiful link with Hodge integrals has been discovered, both by virtual localization [7,10] and by other methods [2]. It is therefore natural to expect that the knowledge of Hurwitz numbers might be used to gain new insights into Hodge integrals. This line of investigations has previously led to a closed-form formula for a generating function for Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,1}$ of curves with one marked point [2,5]. Similarly, in this paper, we consider the following generating function for Hodge integrals over $\overline{\mathcal{M}}_{g,2}$:

$$G(t,k) := \frac{1}{2} + \sum_{g \geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{\mathcal{M}}_{g,2}} \frac{\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)}.$$

For negative integral values of k, we have managed to compute $G(t,k)$ in a closed form by relating the integrals to the almost simple Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$. We then conjecture a simplified version of our rigorously obtained result, and this conjectural counterpart can then be analytically continued to all values of k. We have checked that the conjectural form of our formula holds true for $-60 \leq k \leq 1$, but unfortunately, we have not been able to prove it for arbitrary k. The success of the computation makes us speculate that in more general cases, similar results might be within reach, and the simplicity of the results suggests that new yet undiscovered structures might be present.

This paper is organized as follows: in Section 2, we briefly explain the work of Mednykh and apply it to compute the simple Hurwitz numbers; in Section 3, we find the generating functions for all simple Hurwitz numbers; Section 4 discusses our closed-form formula for the generating function for Hodge integrals over $\overline{\mathcal{M}}_{g,2}$; and, we conclude by drawing the reader's attention to some important open questions.

2. Computations of simple Hurwitz numbers

This section describes our computations of the simple Hurwitz numbers $\tilde{\mu}_{h,n}^{g,n}$. The simple covers of an elliptic curve by elliptic curves are actually unramified, and we obtain the numbers $\tilde{\mu}_{1,n}^{1,n}$ by using the standard theory of two-dimensional lattices (we thank R. Vakil for explaining this approach to us). For other values of g and h, we simplify the general formulas of Mednykh [14] and explicitly compute the numbers for low degrees.

2.1. Unramified covers of a torus by tori

For covers of an elliptic curve by elliptic curves, the Riemann–Hurwitz formula (1.1) becomes

$$r = w - n,$$

but since $n \geq w$, there cannot be any simple branch points and the special point ∞ also has no branching. As a result, the computation for this case reduces to determining the number of degree n unbranched covers of an elliptic curve by elliptic curves. Equivalently, for a given lattice L associated with the target elliptic curve, we need to find the number of inequivalent sublattices $L' \subset L$ of index $[L : L'] = n$. The answer is given by Lemma B.1 to be

$$\tilde{\mu}_{1,n}^{1,n} = \sigma_1(n),$$

where, as usual, $\sigma_k(n) = \sum_{d|n} d^k$. Note that we are doing the actual counting of distinct covers, and our answer $\tilde{\mu}_{1,n}^{1,n}$ is not equal to $\mu_{1,n}^{1,n}$ which is defined by incorporating the automorphism group of the cover differently. This point will become clear in our ensuing discussions.

The generating function for the number of inequivalent simple covers of an elliptic curve by elliptic curves is thus given by

$$\tilde{H}_1^1 = \sigma_1(n)q^n = -\left(\frac{\mathrm{d}\log\eta(q)}{\mathrm{d}t} - \frac{1}{24}\right), \tag{2.1}$$

where $q = \mathrm{e}^t$.

Up to the constant $1/24$, our answer (2.1) is a derivative of the genus-1 free energy $\mathcal{F}_1$ of string theory on an elliptic curve target space. The expression (2.1) can also be obtained by counting distinct orbits of the action of S_n on a set $\mathcal{T}_{n,1,0}$, which will be discussed subsequently. The string theory computation of $\mathcal{F}_1$, however, counts the number $\mu_{1,n}^{1,n} := |\mathcal{T}_{n,1,0}|/n!$ without taking the fixed points of the S_n action into account, and it is somewhat surprising that our counting is related to the string theory answer by simple multiplication by the degree. It turns out that this phenomenon occurs for $g = 1$ because the function $\sigma_1(n)$ can be expressed as a sum of products of $\pi(k)$, where $\pi(k)$ is the number of distinct partitions of the integer k into positive integers, and because this sum precisely appears in the definition of $T_{n,1,0} = |\mathcal{T}_{n,1,0}|$. We will elaborate upon this point in Section 2.3. In other cases, the two numbers $\mu_{h,n}^{g,n}$ and $\tilde{\mu}_{h,n}^{g,n}$ are related by an additive term which generally depends on g, h, and n.

2.2. Low degree computations from the work of Mednykh

The most general Hurwitz enumeration problem for an arbitrary branch type has been formally solved by Mednykh [14]. His answers are based on the original idea of Hurwitz of reformulating the ramified covers in terms of the representation theory of S_n [12]. Let $f : \Sigma_g \to \Sigma_h$ be a degree-n branched cover of a compact connected Riemann surface of

genus-h by a compact connected Riemann surface of genus-g, with r branch points, the orders of whose pre-images being specified by the partitions $\alpha^{(p)} = (1^{t_1^p}, \ldots, n^{t_n^p}) \vdash n$, $p = 1, \ldots, r$. The ramification type of the covering f is then denoted by the matrix $\sigma = (t_s^p)$. Two such branched covers f_1 and f_2 are equivalent if there exists a homeomorphism $\varphi : \Sigma_g \to \Sigma_g$ such that $f_2 = f_1 \circ \varphi$. Now, define the set $\mathcal{B}_{n,h,\sigma}$ as

$$
\mathcal{B}_{n,h,\sigma} = \left\{ (a_1, b_1, \ldots, a_h, b_h, (1^{t_1^1}, \ldots, n^{t_n^1}), \ldots, (1^{t_1^r}, \ldots, n^{t_n^r})) \in (S_n)^{2h+r} \mid \right.
$$
$$
\left. \prod_{i=1}^h [a_i, b_i] \prod_{p=1}^r (1^{t_1^p}, \ldots, n^{t_n^p}) = 1 \right\}, \tag{2.2}
$$

and $\mathcal{T}_{n,h,\sigma} \subset \mathcal{B}_{n,h,\sigma}$ as the subset whose elements generate transitive subgroups of S_n. Then, according to Hurwitz, there is a one-to-one correspondence between irreducible branched covers and elements of $\mathcal{T}_{n,h,\sigma}$. Furthermore, the equivalence relation of covers gets translated into conjugation by a permutation in S_n, i.e. two elements of $\mathcal{T}_{n,h,\sigma}$ are now equivalent iff they are conjugates. Thus, the Hurwitz enumeration problem reduces to counting the number of orbits in $\mathcal{T}_{n,h,\sigma}$ under the action of S_n by conjugation.

Let us denote the orders of the sets by $B_{n,h,\sigma} = |\mathcal{B}_{n,h,\sigma}|$ and $T_{n,h,\sigma} = |\mathcal{T}_{n,h,\sigma}|$. Then, using the classical Burnside's formula, Mednykh obtains the following theorem for the number $N_{n,h,\sigma}$ of orbits:

Theorem 2.1 (Mednykh). *The number of degree-n non-equivalent branched covers of the ramification type $\sigma = (t_s^p)$, for $p = 1, \ldots, r$, and $s = 1, \ldots, n$ is given by*

$$
N_{n,h,\sigma} = \frac{1}{n} \sum_{\substack{\ell \mid v \\ m\ell = n}} \sum_{(1/(t,\ell))\mid d\mid \ell} \frac{\mu(\ell/d) d^{(2h-2+r)m+1}}{(m-1)!}
$$

$$
\times \left[\sum_{j_{k,p}^s} T_{n,h,(s_k^p)} \sum_{x=1}^d \prod_{s,k,p} \left[\frac{\phi(x, s/k)}{d} \right]^{j_{k,p}^s} \prod_{k,p} \binom{s_k^p}{j_{k,p}^1, \ldots, j_{k,p}^{md}} \right], \tag{2.3}
$$

where $t := \mathrm{GCD}\{t_s^p\}$, $v := \mathrm{GCD}\{st_s^p\}$, $(t, \ell) = \mathrm{GCD}(t, \ell)$, $s_k^p = \sum_{s=1}^{md} j_{k,p}^s$, and the sum over $j_{k,p}^s$ ranges over all collections $\{j_{k,p}^s\}$ satisfying the condition

$$
\sum_{\substack{1 \leq k \leq st_s^p/\ell \\ (s/(s,d))\mid k\mid s}} k j_{k,p}^s = \frac{st_s^p}{\ell}
$$

where $j_{k,p}^s$ is non-zero only for $1 \leq k \leq st_s^p/\ell$ and $(s/(s,d))|k|s$. The functions μ and ϕ are the Möbius and von Sterneck functions.

As is apparent from its daunting form, the expression involves many conditional sums and does not immediately yield the desired numerical answers. Mednykh's works, even though quite remarkable, are thus of dormant nature for obtaining the closed-form numerical answers of the Hurwitz enumeration problem. (Recently, closed-form answers for coverings of a Riemann sphere by genus-0,1,2 Riemann surfaces with one non-simple branching have been obtained in [9].)

Interestingly, the general formula (2.3) still has some applicability. For example, in [15], Mednykh considers the special case of branch points whose orders are all equal to the degree of the cover and obtains a simplified formula which is suitable for practical applications. In a similar vein, we discover that for simple branched covers, Mednykh's formula simplifies dramatically and that for some low degrees, we are able to obtain closed-form answers for simple Hurwitz numbers of ramified coverings of genus-h Riemann surfaces by genus-g Riemann surfaces.

2.2.1. The simplifications for simple Hurwitz numbers

We consider degree-n simple branch covers of a genus-h Riemann surface by genus-g Riemann surfaces. A simple branch point has order $(1^{n-2}, 2)$, and thus the branch type is characterized by the matrix $\sigma = (t_s^p)$, for $p = 1, \ldots, r$, and $s = 1, \ldots, n$, where

$$t_s^p = (n-2)\delta_{s,1} + \delta_{s,2}.$$

To apply Mednykh's master formula (2.3), we need to determine $t = \mathrm{GCD}\{t_s^p\}$ and $v = \mathrm{GCD}\{st_s^p\}$, which are easily seen to be

$$t = 1 \quad \text{and} \quad v = \begin{cases} 2 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Because v determines the range of the first sum in the master formula, we need to distinguish when the degree n is odd or even.

2.2.2. Odd degree covers

For degree-n odd, we have $\ell = d = (t, \ell) = 1$ and $m = n$. The constraints $(s/(s,d))|k|s$ and

$$\sum_{1 \leq k \leq st_s^p/\ell} k j_{k,p}^s = \frac{st_s^p}{\ell}$$

then determine the collection $\{j_{k,p}^s\}$ to be

$$j_{k,p}^s = t_s^p \delta_{k,s}.$$

Noting that $\phi(1, 1) = 1$, we see that the master formula now reduces to

$$N_{n,h,\sigma} = \frac{T_{n,h,(s_k^p)}}{n!} \quad (n \text{ odd}), \tag{2.4}$$

where

$$s_k^p = \sum_{s=1}^{n} j_{k,p}^s = t_k^p = (n-2)\delta_{k,1} + \delta_{k,2}. \tag{2.5}$$

2.2.3. Even degree covers

For degree-n even, $v = 2$ and thus $\ell = 1$ or 2.

- $\ell = 1$: The variables take the same values as in the case of n odd, and the $\ell = 1$ contribution to $N_{n,h,\sigma}$ is thus precisely given by (2.4).
- $\ell = 2$: In this case, the summed variables are fixed to be

$$m = \tfrac{1}{2}n, \quad \text{and} \quad d = \ell = 2.$$

Then, one determines that

$$j_{k,p}^s = \tfrac{1}{2}t_1^p \delta_{s,1}\delta_{k,1} + t_2^p \delta_{s,2}\delta_{k,1},$$

from which it follows that

$$\tilde{s}_k^p = \tfrac{1}{2}n\delta_{k,1},$$

where we have put a tilde over s_k^p to distinguish them from (2.5). Using the fact that the number r of simple branch points is even, and the values $\phi(2,1) = \phi(2,2) = -\phi(1,2) = 1$, one can now show that the $\ell = 2$ contribution to $N_{n,h,\sigma}$ is

$$\frac{2^{(h-1)n+1}}{(n/2-1)!} \left(\frac{n}{2}\right)^{r-1} T_{n/2,h,(\tilde{s}_k^p)}.$$

The sum of both contributions is finally given by

$$N_{n,h,\sigma} = \frac{1}{n!}T_{n,h,(s_k^p)} + \frac{2^{(h-1)n+1}}{(n/2-1)!} \left(\frac{n}{2}\right)^{r-1} T_{n/2,h,(\tilde{s}_k^p)} \quad (n \text{ even}). \tag{2.6}$$

Notations. For simple branch types, i.e. for $\sigma = (t_k^p)$ where $t_k^p = (n-2)\delta_{k,1} + \delta_{k,2}$, for $p = 1, \ldots, r$ and $k = 1, \ldots, n$, we will use the notation $T_{n,h,\sigma} =: T_{n,h,r}$.

The computations of fixed-degree-n simple Hurwitz numbers are thus reduced to computing the two numbers $T_{n,h,(s_k^p)}$ and $T_{n/2,h,(\tilde{s}_k^p)}$, only the former being relevant when n is odd. We now compute these numbers for some low degrees and arbitrary genera h and g. The nature of the computations is such that we only need to know the characters of the identity and the transposition elements in S_n.

The term $T_{n/2,h,(\tilde{s}_k^p)}$ can be easily computed:

Lemma 2.1. Let $\tilde{s}_k^p = n\delta_{k,1}$. Then,

$$T_{n,h,(\tilde{s}_k^p)} = n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{n_1+\cdots+n_k=n} \prod_{i=1}^{k} \left[\sum_{\gamma \in \mathcal{R}_{n_i}} \left(\frac{n_i!}{f^\gamma}\right)^{2h-2}\right],$$

where n_i are positive integers, $\mathcal{R}_{n_i}$ the set of all irreducible representations of S_{n_i}, and f^γ the dimension of the representation γ.

For $h = 0$, we can explicitly evaluate this contribution:

Lemma 2.2. *Let* $\tilde{s}_k^p = n\delta_{k,1}$. *Then,*

$$T_{n,0,(\tilde{s}_k^p)} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i > 0}} \binom{n}{n_1, \ldots, n_k} = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n > 1. \end{cases} \quad (2.7)$$

Proof. The first equality follows from the fact that the order of a finite group is equal to the sum of squares of the dimension of its irreducible representations. The second equality follows by noticing that the expression for $T_{n,0,(\tilde{s}_k^p)}/n!$ is the nth coefficient of the formal q-expansion of $\log\left(\sum_{n=0}^\infty q^n/n!\right)$, which is a fancy way of writing q. □

Using (2.4) and (2.6) we have computed closed-form formulas for the simple Hurwitz numbers for arbitrary source and target Riemann surfaces for degrees less than 8 in Appendix C.

2.3. Cautionary remarks

Hurwitz numbers are sometimes *defined* to be $T_{n,h,\sigma}/n!$, counting orbits as if there were no fixed points of the action S_n on $\mathcal{T}_{n,h,\sigma}$. The master formula obtained by Mednykh uses the Burnside's formula to account for the fixed points. In the case of simple Hurwitz numbers, this will lead to an apparent discrepancy between our results and those obtained by others for even degree covers, the precise reason being that for even degree covers, say of degree-$2n$, the action of $(2^n) \in S_{2n}$ on $\mathcal{T}_{2n,h,\sigma}$ has fixed points which are counted by the second term in (2.6). Consequently, to obtain the usual even degree Hurwitz numbers, we just need to consider the contribution of the first term in (2.6). For odd degree cases, there is no non-trivial fixed points, and our formula needs no adjustment. The following examples of the discussion would be instructive:

2.3.1. Example 1

Let us explicitly count the double covers of an elliptic curve by genus-g Riemann surfaces. The set $\mathcal{T}_{2,1,2g-2}$ is given by

$$\mathcal{T}_{2,1,2g-2} = \{(a, b, (2)^{2g-2}) \in S_2^{2g} | aba^{-1}b^{-1}(2)^{2g-2} = 1\}.$$

Since S_2 is commutative and $(2)^2 = 1$, any pair $(a, b) \in S_2 \times S_2$ satisfies the required condition. Hence, the order of $\mathcal{T}_{2,1,2g-2}$ is four. Now, to count non-equivalent coverings, we need to consider the action of S_2 on the set $\mathcal{T}_{2,1,2g-2}$ by conjugation. Again, since S_2 is Abelian, it is clear that it acts trivially on the set and thus that there are four inequivalent double covers of an elliptic curve by genus-g Riemann surfaces. The commonly adopted

definition of Hurwitz number, however, specifies that we should take the order of the set $\mathcal{T}_{2,1,2g-2}$ and divide it by the dimension of S_2, yielding 2 as its answer. This number 2 is precisely the first contribution in the Burnside's formula:

$$N_{2,1,2g-2} = \frac{1}{|S_2|} \sum_{\sigma \in S_2} |F_\sigma| = \frac{F_{(1^2)}}{2!} + \frac{F_{(2)}}{2!} = 2 + 2 = 4,$$

where $|F_\sigma|$ is the order of the fixed-point set under the action of $\sigma \in S_2$. For odd n, S_n acts freely on the set $\mathcal{T}_{n,h,r}$, but for even n, it has fixed points and our formula (2.6) accounts for the phenomenon, truly *counting* the number of inequivalent covers.

To avoid possible confusions, we thus use the following notations to distinguish the two numbers:

$$\mu_{h,n}^{g,n} := \frac{T_{n,h,r}}{n!} \quad \text{for all } n,$$

and

$$\tilde{\mu}_{h,n}^{g,n} := N_{n,h,r}.$$

It turns out that current researchers are mostly interested in $\mu_{h,n}^{g,n}$; for example, it is this definition of simple Hurwitz numbers that appears in the string theory literature and in relation to Gromov–Witten invariants. In this paper, we will compute the numbers $\tilde{\mu}_{h,n}^{g,n}$ and indicate the $\ell = 2$ contributions which can be subtracted to yield $\mu_{h,n}^{g,n}$. We will however find generating functions only for the case $\mu_{h,n}^{g,n}$.

2.3.2. Example 2

The above discussion shows that the two numbers $\tilde{\mu}_{h,n}^{g,n}$ and $\mu_{h,n}^{g,n}$ differ by the second term in (2.6) and thus are not related by simple multiplicative factors. For $h = 1$ and $g = 1$, however, we have previously observed that $\tilde{H}_1^1$ given in (2.1) is equal to $\partial_t \mathcal{F}_1$, up to an additive constant, implying that

$$\tilde{\mu}_{1,n}^{1,n} = n\mu_{1,n}^{1,n}.$$

This special equality actually follows from Lemma B.2, since we know that $\tilde{\mu}_{1,n}^{1,n} = \sigma_1(n)$ and since one can show that

$$\mu_{1,n}^{1,n} := \frac{T_{n,1,0}}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{m_1+\cdots+m_k=n} \left(\prod_{i=1}^{k} \pi(m_i) \right).$$

2.4. Recursive solutions to $T_{n,h,r}$ for an elliptic curve ($h = 1$)

Elliptic curve is the simplest Calabi–Yau manifold and is of particular interest to string theorists. The free energies $\mathcal{F}_g$ count the numbers $\mu_{h,n}^{g,n}$, and string theorists have computed $\mathcal{F}_g$ for $g \leq 8$ [16]. Using the approach described in the previous subsection, we have obtained the closed-form formulas for $N_{n,h,r}$ for $n < 8$. For $h = 1$, its $\ell = 1$ parts agree with the known free energies $\mathcal{F}_g$. Although our results are rewarding in that they

give explicit answers for all g and h, further computations become somewhat cumbersome beyond degree 8. For higher degrees, we therefore adopt a recursive method to solve $T_{n,1,r}$ on a case-by-case basis.

The number of reducible covers $B_{n,h,\sigma}/n!$ and that of irreducible covers $T_{n,h,\sigma}/n!$ are related by exponentiation [15]:

$$\sum_{\sigma \geq 0} \frac{B_{n,h,\sigma}}{n!} w^\sigma = \exp\left(\sum_{\sigma \geq 0} \frac{T_{n,h,\sigma}}{n!} w^\sigma\right), \tag{2.8}$$

where w^σ denotes the multi-product

$$w^\sigma := \prod_{p=1}^{r} \prod_{k=1}^{n} w_{pk}^{t_k^p}$$

in the indeterminates w_{pk} and $\sigma \geq 0$ means $t_k^p \geq 0 \ \forall p, k$. From (2.8), one can derive

$$B_{n,h,\sigma} = \sum_{k=1}^{n} \frac{1}{k!} \binom{n}{n_1, \ldots, n_k} \sum_{\substack{n_1 + \cdots + n_k = n \\ \sigma_1 + \cdots + \sigma_k = \sigma}} T_{n_1,h,\sigma_1} \cdots T_{n_k,h,\sigma_k}.$$

In particular, for simple covers of an elliptic curve, partitioning σ appropriately yields

$$\frac{T_{n,1,r}}{n!} = \frac{B_{n,1,r}}{n!} - \sum_{k=2}^{n} \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ 2\ell_1 + \cdots + 2\ell_k = r}} \binom{r}{2\ell_1, \ldots, 2\ell_k} \prod_{i=1}^{k} \frac{T_{n_i,1,2\ell_i}}{n_i!}, \tag{2.9}$$

where n_i and ℓ_i are positive and non-negative integers, respectively. For fixed degree n, (2.9) expresses $T_{n,1,r}$ in terms of lower degree and lower genera Hurwitz numbers, and $B_{n,1,r}$. The number $B_{n,1,r}$ in this case reduces to

$$B_{n,1,r} = \frac{n!}{2^r} \sum_{k=1}^{n} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1 \geq n_2 \geq \cdots \geq n_k}} \left[\sum_{i \in \mathcal{I}} p_i(p_i - 1) \prod_{j \neq i} \left(\frac{p_i - 2 - p_j}{p_i - p_j}\right)\right]^r,$$

where $p_i = n_i + k - i$ and $\mathcal{I} = \{i \in \{1, \ldots, k\} | (p_i - 2) \geq 0\}$. In Appendix D, we provide the explicit values of $B_{n,h,r}$ for $n \leq 9$.

We have implemented the recursion into a *Mathematica* program which, using our results from the previous subsection as inputs, computes $T_{n,1,r}$ for $n \geq 8$. For the sake of demonstration, we present some numerical values of $T_{n,1,r}/n!$ for $n \leq 9$ in Appendix E.

3. Generating functions for simple Hurwitz numbers

Recently, Göttsche has conjectured an expression for the generating function for the number of nodal curves on a surface S, with a very ample line bundle L, in terms of certain universal power series and basic invariants [8]. More precisely, he conjectures that the generating function $T(S, L)$ for the number of nodal curves may have the form

$$T(S, L) = \exp(c_2(S)A + K_S^2 B + K_S \cdot LC + L^2 D),$$

where A, B, C, D are universal power series in some formal variables and K_S the canonical line bundle of S.

In a kindred spirit, it would be interesting to see whether such universal structures exist for Hurwitz numbers. For a curve, the analogues of K_S and $c_2(S)$ would be the genus of the target and L the degree of the branched cover. It turns out that for simple Hurwitz numbers, we are able to find their generating functions in closed-forms, but the resulting structure is seen to be more complicated than that for the case of surfaces.

3.1. Summing up the string coupling expansions

The free energies $\mathcal{F}_g$ on an elliptic curve have been computed in [16] up to $g = 8$, and their q-expansions agree precisely with our results shown in Appendix E. (Here, $q = \exp(\hat{t})$, where $\hat{t}$ is a formal variable dual to the Kähler class of the elliptic curve.)

For a fixed degree $n < 8$, we know $\mathcal{F}_g$ for all g, so we can sum up the expansion

$$\mathcal{F} = \sum_g \lambda^{2g-2} \mathcal{F}_g, \tag{3.1}$$

up to the given degree n in the world-sheet instanton expansion. That is, we are summing up the string coupling expansions, and this computation is a counterpart of "summing up the world-sheet instantons" which string theorists are accustomed to studying.

Consider the following generating function for simple Hurwitz numbers:

$$\Phi(h) = \sum_{g,n} \frac{T_{n,h,r}}{n!} \frac{\lambda^r}{r!} q^n = \sum_{g,n} \mu_{h,n}^{g,n} \frac{\lambda^r}{r!} q^n, \tag{3.2}$$

which coincides with the total free energy (3.1) for $h = 1$. For low degree simple covers of an elliptic curve, we can use our results (E.1) to perform the summation over the number r of simple branch points and get

$$\Phi(1) = \sum_g \lambda^{2g-2} \mathcal{F}_g = -\log(q^{-1/24}\eta(q)) + 2[\cosh(\lambda) - 1]q^2$$

$$+ 2[\cosh(3\lambda) - \cosh(\lambda)]q^3$$

$$+ 2\left[\cosh(6\lambda) + \frac{1}{2}\cosh(2\lambda) - \cosh(3\lambda) + \cosh(\lambda) - \frac{3}{2}\right]q^4 + \mathcal{O}(q^5).$$

The partition function $Z = e^{\Phi(1)}$ is then given by

$$Z = 1 + q + 2\cosh(\lambda)q^2 + [1 + 2\cosh(3\lambda)]q^3 + [1 + 2\cosh(2\lambda) + 2\cosh(6\lambda)]q^4$$
$$+ [1 + 2\cosh(2\lambda) + 2\cosh(5\lambda) + 2\cosh(10\lambda)]q^5 + \mathcal{O}(q^6).$$

At this point, we can observe a pattern emerging, and indeed, the partition function can be obtained to all degrees from the following statement which, we subsequently discovered, was also given in [1]:

Claim 3.1. *The partition function Z, or the exponential of the generating function for simple Hurwitz numbers, for an elliptic curve target is given by*

$$Z = 1 + q + \sum_{n \geq 2} \left(\sum_{\gamma \in \mathcal{R}_n} \cosh\left[\binom{n}{2} \frac{\chi_\gamma(2)}{f^\gamma} \lambda \right] \right) q^n. \tag{3.3}$$

Proof. From (2.9), we see that

$$\frac{B_{n,1,r}}{n!r!} = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ 2\ell_1 + \cdots + 2\ell_k = r}} \prod_{i=1}^{k} (\mathcal{F}_{\ell_i+1})_{q^{n_i}}, \tag{3.4}$$

where, as before, n_i and ℓ_i are positive and non-negative integers, respectively, and $(\mathcal{F}_g)_{q^m}$ is the coefficient of q^m in the genus-g free energy. The numbers $B_{n,1,r}$ are determined to be

$$B_{n,1,r} = \begin{cases} n! \binom{n}{2}^r \left[\sum_{\gamma \in \mathcal{R}_n} \left(\frac{\chi_\gamma(2)}{f^\gamma} \right)^r \right] & \text{for } n \geq 2, \\ \delta_{r,0} & \text{for } n \leq 1. \end{cases}$$

Now, multiplying both sides of (3.4) by $\lambda^r q^n$ and summing over all even $r \geq 0$ and all $n \geq 0$ proves the claim. $\qquad \square$

The argument of hyperbolic-cosine is known as the central character of the irreducible representation γ and can be evaluated as in (A.1).

3.1.1. Further recursions for closed-form answers

The above explicit form of the partition function gives rise to a powerful way of recursively solving for the simple Hurwitz numbers $\mu_{1,n}^{g,n}$ for a given degree n, similar to those given in (E.1). Let us consider this more closely. Suppose that, knowing closed-form formulas for $\mu_{1,n_i}^{g_i,n_i}$ for all $n_i < n$ and arbitrary g_i, we are interested in deriving a closed-form formula for $\mu_{1,n}^{g,n}$, where g is again arbitrary. The key idea is to match the coefficient of $\lambda^{2g-2}q^n$ in the expansion of the partition function Z with the coefficient of the same term in the expansion

$$\exp[\Phi(1)] = 1 + \Phi(1) + \frac{1}{2}[\Phi(1)]^2 + \cdots + \frac{1}{k!}[\Phi(1)]^k + \cdots.$$

The coefficient of $\lambda^{2g-2}q^n$ in $\Phi(1)$ contains precisely what we are looking for, namely $\mu_{1,n}^{g,n}$. On the other hand, the coefficients of $\lambda^{2g-2}q^n$ in $[\Phi(1)]^k$, for $k > 1$, are given in terms of $\mu_{1,n_i}^{g_i,n_i}$, where $n_i < n$ and $g_i \leq g$. But, by hypothesis, we know $\mu_{1,n_i}^{g_i,n_i}$ for all $n_i < n$, and therefore we can solve for $\mu_{1,n}^{g,n}$ in a closed-form. Using this method, we have obtained the degree-8 Hurwitz numbers, and the answer agrees with the known results as well as the computation done by our earlier recursive method.

This recursive method also works for determining the general simple Hurwitz numbers $\mu_{h,n}^{g,n}$, upon using the general "partition function" (3.5) in place of Z.

3.2. The generating functions for target curves of arbitrary genus

For arbitrary genus targets, there is a natural generalization of the above discussion on the generating functions. We have previously defined the generating function $\Phi(h)$ to be

$$\Phi(h) = \sum_{r,n \geq 0} \mu_{h,n}^{g,n} \frac{\lambda^r}{r!} q^n,$$

and seen that for $h = 1$, it coincides with the total free energy of string theory on an elliptic curve target, where λ is identified with the string coupling constant. For $h \neq 1$, however, the formal parameter λ should be actually viewed as the parameter $\hat{t}_1^1$ dual to the first descendant of the Kähler class. (Unfortunately, we have previously used the notation t_k^p to denote the branching matrix. Here, to avoid confusions, we use $\hat{t}$ for the coordinates that appear in the Gromov–Witten theory.) We do not need an extra genus-keeping parameter, because for simple covers of a fixed target space with a given number of marked points r, choosing the degree of the map fixes the genus of the source Riemann surface uniquely. For the purpose of finding a nice generating function, it is thus convenient to treat r and n as independent indices, with the requirement that they be both non-negative.

For $r = 0$, our previous computations of the simple Hurwitz numbers need to be modified as

$$\frac{T_{n,h,0}}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i > 0}} \prod_{i=1}^{k} (n_i)!^{2h-2} \left[\sum_{\gamma \in \mathcal{R}_{n_i}} (f^\gamma)^{2-2h} \right].$$

Also, note that $N_{1,h,r} = \delta_{r,0}$. Then, we have

Claim 3.2. *The generalized "partition function" $Z(h) = \exp(\Phi(h))$ for all h is given by*

$$Z(h) = 1 + q + \sum_{n \geq 2} \sum_{\gamma \in \mathcal{R}_n} \left(\frac{n!}{f^\gamma} \right)^{2h-2} \cosh \left[\binom{n}{2} \frac{\chi_\gamma(2)}{f^\gamma} \lambda \right] q^n. \tag{3.5}$$

Proof. The proof is exactly the same as that of Claim 3.1. One just needs to keep track of extra factors in the general form of $B_{n,h,r}$. For genus $h = 0$, when applying the Riemann–Hurwitz formula, we must remember to use the correctly defined arithmetic genus

of reducible curves and, as a result, sum over all even $r \geq 0$ in $B_{n,0,r}$; doing so takes into account the degree-1 covers in the exponential. $\qquad \square$

4. Hodge integrals on $\bar{\mathcal{M}}_{g,2}$ and Hurwitz numbers

In the modern language of Gromov–Witten theory, the simple Hurwitz numbers are equal to

$$\mu_{h,n}^{g,n} := \frac{T_{n,h,r}}{n!} = \langle \tau_{1,1}^r \rangle_{g,n},$$

where $r = 2(1 - h)n + 2(g - 1)$ and $\tau_{k,1}$ is the kth descendant of the Kähler class of the target genus-h Riemann surface. We can organize these numbers into a generating function as follows:

$$H_h^g := \sum_n \frac{1}{r!} \langle \tau_{1,1}^r \rangle_{g,n} (\hat{t}_1^1)^r \, \mathrm{e}^{n\hat{t}} = \sum_n \frac{1}{r!} \frac{T_{n,h,r}}{n!} (\hat{t}_1^1)^r \, \mathrm{e}^{n\hat{t}}, \tag{4.1}$$

where $\hat{t}_1^1$ and $\hat{t}$ are coordinates dual to $\tau_{1,1}$ and $\tau_{0,1}$, respectively. In this paper, we have determined (4.1) for all g and h up to degree $n = 7$.

For $h = 0$ and $h = 1$, these generating functions arise as genus-g free energies of string theory on $\mathbb{P}^1$ and an elliptic curve as target spaces, respectively, evaluated by setting all coordinates to zero except for $\hat{t}_1^1$ and $\hat{t}$. For definitions of Hodge integrals, see [5,6].

4.1. Generating functions for Hodge integrals

The Hurwitz enumeration problem has been so far investigated intensely mainly for branched covers of the Riemann sphere. In this case, the almost simple Hurwitz numbers for covers with one general branch point can be expressed explicitly in terms of certain Hodge integrals. An interesting application of this development is to use the generating function for Hurwitz numbers $\mu_{0,1}^{g,d}(d)$ to derive a generating function for Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,1}$. More precisely, the formula

$$F(t, k) := 1 + \sum_{g \geq 1} t^{2g} \sum_{i=0}^g k^i \int_{\overline{\mathcal{M}}_{g,1}} \psi^{2g-2+i} \lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)} \right)^{k+1}, \tag{4.2}$$

which was first obtained by Faber and Pandharipande [5] by using virtual localization techniques has been rederived by Ekedahl et al. [2] by using the generating function for Hurwitz numbers for branched covers whose only non-simple branch point has order equal to the degree of the cover.

In this paper, we speculate a possible connection between the Hurwitz numbers for $\mathbb{P}^1$ and generating functions for Hodge integrals on $\overline{\mathcal{M}}_{g,n}$, $n \geq 1$. For this purpose, let us rewrite $F(t, k)$ as

$$F(t, k) = 1 + \sum_{g \geq 1} t^{2g} \sum_{i=0}^g k^i \int_{\bar{\mathcal{M}}_{g,1}} \frac{\lambda_{g-i}}{1 - \psi_1}. \tag{4.3}$$

Now, recall that the simple Hurwitz numbers $\mu_{0,n}^{g,n}(1^n)$, henceforth abbreviated $H_{g,n}$, have the following Hodge integral expression [7]:

$$H_{g,n} := \mu_{0,n}^{g,n} = \frac{(2g-2+2n)!}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{\prod_{i=1}^n(1-\psi_i)} \tag{4.4}$$

for $(g,n) \neq (0,1), (0,2)$. The degree-1 simple Hurwitz numbers are $H_{g,1} = \delta_{g,0}$, thus (4.4) yields the relation

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{1-\psi_1} = 0 \quad \text{for } g \geq 1,$$

which implies from (4.3) that $F(t, -1) = 1$, in accord with the known answer (4.2). Naively, we thus see that the simple Hurwitz numbers are coefficients of $F(t, k)$ evaluated at special k.

In a similar spirit, we can speculate a crude generating function for Hodge integrals with two marked points:

$$G(t, k) = \frac{1}{2} + \sum_{g \geq 1} t^{2g} \sum_{i=0}^g k^i \int_{\overline{\mathcal{M}}_{g,2}} \frac{\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)}.$$

Our goal is to find a closed-form expression for this generating function $G(t, k)$. Without much work, we can immediately evaluate $G(t, k)$ at certain special values of k:

Claim 4.1. *The generating function $G(t, k)$ can be evaluated at $k = -1$ to be*

$$G(t, -1) = \frac{1}{2} - \frac{1}{t^2}\left(\cos t + \frac{t^2}{2} - 1\right) = \frac{1}{2}\left(\frac{\sin(t/2)}{t/2}\right)^2, \tag{4.5}$$

and similarly at $k = 0$ to be

$$G(t, 0) = \frac{1}{2}\left(\frac{t}{\sin t}\right) = \frac{1}{2}\frac{t/2}{\sin(t/2)}\frac{1}{\cos(t/2)}. \tag{4.6}$$

Proof. At $k = -1$, we can use (4.4) to get

$$G(t, -1) = \sum_{g \geq 0} (-1)^g \frac{2t^{2g}}{(2g+2)!} H_{g,2}. \tag{4.7}$$

We have previously computed $H_{g,2} = N_{2,0,2g+2}/2 = 1/2$, and we can then perform the summation in (4.7) and get the desired result. To evaluate $G(t, 0)$, we use the following λ_g-conjecture, which has been recently proven by Faber and Pandharipande [6]:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\lambda_g = \binom{2g+n-3}{\alpha_1,\ldots,\alpha_n}\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g.$$

One can now compute

$$\int_{\overline{\mathcal{M}}_{g,2}} \frac{\lambda_g}{(1 - \psi_1)(1 - \psi_2)} = \frac{(2^{2g-1} - 1)}{(2g)!} |B_{2g}|,$$

and obtain the result. □

To extract the terms without λ_k insertions, consider the scaling limit

$$G(tk^{1/2}, k^{-1}) = \frac{1}{2} + \sum_{g \geq 1} t^{2g} \sum_{i=0}^{g} k^{g-i} \int_{\overline{\mathcal{M}}_{g,2}} \frac{\lambda_{g-i}}{(1 - \psi_1)(1 - \psi_2)} \xrightarrow{k \to 0} \frac{1}{2}$$

$$+ \sum_{g \geq 1} t^{2g} \sum_{i=0}^{g} \int_{\overline{\mathcal{M}}_{g,2}} \frac{1}{(1 - \psi_1)(1 - \psi_2)}. \tag{4.8}$$

The asymptotic behavior (4.8) can be explicitly evaluated as follows:

Claim 4.2. *The asymptotic limit of $G(t, k)$ is*

$$G(tk^{1/2}, k^{-1}) \xrightarrow{k \to 0} \frac{\exp(t^2/3)}{2t} \sqrt{\pi} \, \mathrm{Erf} \left[\frac{t}{2} \right],$$

and thus, the integrals can be evaluated to be

$$\int_{\overline{\mathcal{M}}_{g,2}} \frac{1}{(1 - \psi_1)(1 - \psi_2)} = \frac{1}{2} \sum_{m=0}^{g} \frac{1}{m! 12^m} \frac{(g - m)!}{(2g - 2m + 1)!}.$$

Proof. This is an easy consequence of the following Dijkgraaf's formula which appeared in the work of Faber [3]:

$$\langle \tau_0 \tau(w) \tau(z) \rangle = \exp \left(\frac{(w^3 + z^3) \hbar}{24} \right) \sum_{n \geq 0} \frac{n!}{(2n + 1)!} \left[\frac{1}{2} wz(w + z) \hbar \right]^n, \tag{4.9}$$

where $\tau(w) = \sum_{n \geq 0} \tau_n w^n$ and $\hbar$ is a formal genus-expansion parameter defined by

$$\langle \rangle = \sum_{g \geq 0} \langle \rangle_g \hbar^g.$$

Setting $w = z = \hbar^{-1} = t$ in (4.9) and noting that

$$\sum_{n \geq 0} \frac{1}{(2n + 1)!!} t^{2n+1} = e^{t^2/2} \sqrt{\frac{\pi}{2}} \, \mathrm{Erf} \left[\frac{t}{\sqrt{2}} \right]$$

gives the result, upon using the string equation on the left-hand side. □

For future reference, it would be desirable to find an explicit series expansion of $G(t, k)$. Using Faber's Maple program for computing the intersection numbers on $\mathcal{M}_{g,n}$ [4], the

generating function can be seen to have an expansion of the form

$$
G(t,k) = \frac{1}{2} + \left(\frac{1}{12} + \frac{1}{8}k \right) t^2 + \left(\frac{7}{720} + \frac{73}{2880}k + \frac{49}{2880}k^2 \right) t^4
$$
$$
+ \left(\frac{31}{30\,240} + \frac{253}{72\,576}k + \frac{983}{241\,920}k^2 + \frac{1181}{725\,760}k^3 \right) t^6 + \mathcal{O}(t^8). \qquad (4.10)
$$

4.2. Relation to Hurwitz numbers $\mu_{h,2}^{g,2k}(k,k)$

We now relate the generating function $G(t,k)$ to the Hurwitz numbers $\mu_{h,2}^{g,2k}(k,k)$, which we are able to compute explicitly. This connection allows us to evaluate $G(t,k)$ for all $k \in \mathbb{Z}_{<0}$. From the work of [2], we know that

$$
\mu_{0,2}^{g,2k}(k,k) = \frac{(2k+2g)!}{2} \frac{k^{2k}}{(k!)^2} \int_{\mathcal{M}_{g,2}} \frac{c(\Lambda_{g,2}^\vee)}{(1-k\psi_1)(1-k\psi_2)},
$$

which we can rewrite as

$$
\mu_{0,2}^{g,2k}(k,k) = \frac{(2k+2g)!}{2} \frac{k^{2k+2g-1}}{(k!)^2} \sum_{i=0}^{g} k^i \int_{\mathcal{M}_{g,2}} \frac{(-1)^{g-i}\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)}.
$$

This implies that for integers $k > 0$,

$$
G(it,-k) = \frac{1}{2} + \sum_{g \ge 1} \frac{2t^{2g}}{(2k+2g)!} \frac{(k!)^2}{k^{2k+2g-1}} \mu_{0,2}^{g,2k}(k,k). \qquad (4.11)
$$

By using the expansion (4.10) and matching coefficients with (4.11), one can thus obtain the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$. We have listed the numbers for $g \le 4$ in Appendix F.

It is in fact possible to determine the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$ from the work of Shapiro et al. on enumeration of edge-ordered graphs [17]. According to Theorem 9 of their paper, (Actually, their formula has a minor mistake for the case when $n = 2k$ is partitioned into (k,k) for odd genus. More precisely, when the summation variable s in their formula equals $(g+1)/2$, for an odd genus g, there is a symmetry factor of $1/2$ in labeling the edges because the two disconnected graphs are identical except for the labels.) the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$ are given by

$$
\mu_{0,2}^{g,2k}(k,k) = N(2k, 2k+2g, (k,k))
$$
$$
- \binom{2k}{k} \frac{(2k+2g)!}{(2k)!} k^{2k-2+2g} \frac{1}{2} \left[\sum_{s=0}^{g+1} \delta_{2s}^k \delta_{2g+2-2s}^k \right], \qquad (4.12)
$$

where the numbers δ_{2g}^k are defined by

$$
\sum_{g=0}^{\infty} \delta_{2g}^k t^{2g} = \left(\frac{\sinh(t/2)}{t/2} \right)^{k-1},
$$

and can be written explicitly as

$$\delta_{2g}^k = \frac{1}{(k+2g-1)!} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \left(\frac{k-1}{2} - m\right)^{k+2g-1}.$$

The number $N(2k, 2k+2g, (k, k))$, which counts the number of certain edge-ordered graphs, is given by

$$N(2k, 2k + 2g, (k, k)) = \frac{|C(k,k)|}{[(2k)!]^2} \sum_{\rho \vdash 2k} f^\rho (h(\rho') - h(\rho))^{2k+2g} \chi_\rho(k, k), \qquad (4.13)$$

where $|C(k,k)|$ is the order of the conjugacy class $C(k,k)$, ρ' is the partition conjugate to ρ, and $h(\rho) = \sum_i^m (i-1)\rho_i$ for $\rho = (\rho_1, \ldots, \rho_m) \vdash 2k$. Hence, the problem of finding $\mu_{0,2}^{g,2k}(k, k)$ reduces down to evaluating (4.13).

Claim 4.3. *For $k \geq 2$,*

$$N(2k, 2k + 2g, (k, k)) = \frac{(k-1)!}{2k \cdot k!(2k)!} \left\{ \frac{2[k(k-2)]^{2g+2k}(2k)!}{k!(1+k)!} + \sum_{m=0}^{k-1} \binom{2k-1}{m} (-1)^m \right.$$

$$\times [k(2k - 2m - 1)]^{2k+2g} + \sum_{m=k}^{2k-1} \binom{2k-1}{m} (-1)^{m-1}$$

$$\times [k(2k - 2m - 1)]^{2k+2g} + 2 \sum_{m=0}^{k-3} \sum_{p=1}^{k-m-1} \binom{k-1}{m}\binom{k-1}{m+p}$$

$$\left. \times \frac{p^2}{k^2 - p^2} \frac{(2k)!}{(k!)^2} (-1)^{p+1} [k(k - 2m - p - 1)]^{2k+2g} \right\}.$$

Proof. To each irreducible representation labeled by $\rho = (\rho_1, \ldots, \rho_j) \vdash 2k$, we can associate a Young diagram with j rows, the ith row having length ρ_i. According to the *Murnaghan–Nakayama* rule, the diagram corresponding to an irreducible representation ρ for which $\chi_\rho(k, k) \neq 0$, must be either (a) a hook or (b) a union of two hooks. After long and tedious computations, we arrive at the following results:

(a) There are $2k$ "one-hook" diagrams.
   (i) The diagram with leg-length m for $0 \leq m \leq k-1$ gives

   $$f^\rho = \binom{2k-1}{m}, \quad \chi_\rho(k, k) = (-1)^m, \quad h(\rho') - h(\rho) = k(2k - 2m - 1).$$

   (ii) The diagram with leg-length m for $k \leq m \leq 2k - 1$ gives

   $$f^\rho = \binom{2k-1}{m}, \quad \chi_\rho(k, k) = (-1)^{m-1},$$

   $$h(\rho') - h(\rho) = k(2k - 2m - 1).$$

(b) There are $k(k-1)/2$ "two-hook" diagrams.

(i) For each value of m and p satisfying $0 \leq m \leq k-3$ and $1 \leq p \leq k-m-1$, respectively, there is a diagram with $k-m$ columns and $p+m+1$ rows. Such diagram has

$$f^\rho = \binom{k-1}{m}\binom{k-1}{m+p}\frac{p^2}{k^2-p^2}\frac{(2k)!}{(k!)^2}, \qquad \chi_\rho(k,k) = 2(-1)^{p+1},$$

$$h(\rho') - h(\rho) = k(k-2m-p-1).$$

(ii) One diagram has two columns and k rows. It corresponds to the irreducible representation with

$$f^\rho = \frac{(2k)!}{k!(k+1)!}, \qquad \chi_\rho(k,k) = 2, \qquad h(\rho') - h(\rho) = k(k-2).$$

Furthermore, after some simple combinatorial consideration, we find that $|C(k,k)| = (2k)!(k-1)!/(2k \cdot k!)$. Finally, substituting in (4.13) the values of f^ρ, $\chi_\rho(k,k)$ and $h(\rho') - h(\rho)$ for the above $k(k+3)/2$ irreducible representations gives the desired result.   □

By using (4.11) and (4.12), we can now rewrite $G(it, -k)$ as

Claim 4.4. *For integral $k \geq 2$,*

$$G(it,-k) = \frac{2(k-1)!}{(k+1)!t^{2k}} \cosh\left[(k-2)t\right] + \frac{2(k!)^2}{k(2k)!t^{2k}} \sum_{m=0}^{k-1} \binom{2k-1}{m}(-1)^m$$

$$\times \cosh\left[(2k-2m-1)t\right] + \frac{2}{kt^{2k}} \sum_{m=0}^{k-3}\sum_{p=1}^{k-m-1} \binom{k-1}{m}\binom{k-1}{m+p}$$

$$\times \frac{p^2}{k^2-p^2}(-1)^{p+1}\cosh\left[(k-2m-p-1)t\right] - \frac{1}{kt^2}\left(\frac{\sinh[t/2]}{(t/2)}\right)^{2k-2}.$$

(4.14)

Proof. By substituting the expression (4.12) into (4.11) and summing over the δ terms, we get

$$G(it,-k) = \frac{1}{2} + \sum_{g\geq 1} \frac{2(k!)^2 t^{2g}}{(2k+2g)!k^{2k+2g-1}} N(2k, 2k+2g, (k,k))$$

$$- \frac{1}{kt^2}\left(\frac{\sinh(t/2)}{t/2}\right)^{2k-2} + \frac{\delta_0^k \delta_0^k}{kt^2} + \frac{2}{k}\delta_0^k \delta_2^k$$

$$= \frac{1}{2} + \sum_{\ell \geq 0} \frac{2(k!)^2 t^{2\ell-2k}}{(2\ell)! k^{2\ell-1}} N(2k, 2\ell, (k,k)) - \sum_{\ell \geq 0}^{k} \frac{2(k!)^2 t^{2\ell-2k}}{(2\ell)! k^{2\ell-1}} N(2k, 2\ell, (k,k))$$

$$- \frac{1}{kt^2} \left(\frac{\sinh(t/2)}{t/2} \right)^{2k-2} + \frac{\delta_0^k \delta_0^k}{kt^2} + \frac{2}{k} \delta_0^k \delta_2^k. \tag{4.15}$$

But, by Lemma B.3, $N(2k, 2\ell, (k,k)) = 0$ for $\ell \leq k-2$. Furthermore, we have

$$- \frac{2(k!)^2}{(2k)! k^{2k-1}} N(2k, 2k, (k,k)) + \frac{2}{k} \delta_0^k \delta_2^k = - \frac{2(k!)^2}{(2k)! k^{2k-1}} \mu_{0,2}^{0,2k}(k,k) = -\frac{1}{2},$$

and

$$- \frac{2(k!)^2}{(2k-2)! k^{2k-3} t^2} N(2k, 2k-2, (k,k)) + \frac{\delta_0^k \delta_0^k}{kt^2} \propto N_c(2k, 2k-2, (k,k)) = 0,$$

where we have used the known fact [17] that

$$\mu_{0,2}^{0,2k}(k,k) = \binom{2k}{k} \frac{k^{2k-1}}{4}.$$

and Lemma B.4. Thus, we have

$$G(it, -k) = \sum_{\ell \geq 0} \frac{2(k!)^2 t^{2\ell-2k}}{(2\ell)! k^{2\ell-1}} N(2k, 2\ell, (k,k)) - \frac{1}{kt^2} \left(\frac{\sinh(t/2)}{t/2} \right)^{2k-2},$$

where the first term can now be easily summed to yield our claim. $\quad\square$

It turns out that there are some magical simplifications, and we find for a few low values of k that

$$G(t, -1) = \frac{1}{2} \left(\frac{\sin(t/2)}{t/2} \right)^2, \qquad G(t, -2) = \frac{1}{6} [2 + \cos(t)] \left(\frac{\sin(t/2)}{t/2} \right)^4,$$

$$G(t, -3) = \frac{1}{30} [8 + 6 \cos(t) + \cos(2t)] \left(\frac{\sin(t/2)}{t/2} \right)^6,$$

$$G(t, -4) = \frac{1}{140} [32 + 29 \cos(t) + 8 \cos(2t) + \cos(3t)] \left(\frac{\sin(t/2)}{t/2} \right)^8,$$

and so forth. We have explicitly computed $G(t, -k)$ for $k \leq 60$, and based on these computations, we conjecture the following general form:

Conjecture 4.1. *For integers $k \geq 1$, the generating function is given by*

$$G(t, -k) = \frac{2(k-1)! k!}{(2k)!} \left(\frac{\sin(t/2)}{t/2} \right)^{2k} \left[2^{2(k-2)+1} + \sum_{n=1}^{k-1} \left[\sum_{i=0}^{k-n-1} \binom{2k-1}{i} \right] \cos(nt) \right].$$

Let us rewrite the summation as follows:

$$\sum_{n=1}^{k-1}\left[\sum_{i=0}^{k-n-1}\binom{2k-1}{i}\right]\cos(nt) = \sum_{\ell=0}^{k-2}\binom{2k-1}{\ell}\left(\sum_{n=1}^{k-1-\ell}\cos(nt)\right)$$

$$= \frac{1}{2}\sum_{\ell=0}^{k-2}\binom{2k-1}{\ell}\left[\frac{\sin[(2k-1-2\ell)t/2]}{\sin(t/2)}-1\right].$$

(4.16)

The last expression in (4.16) can now be explicitly summed, leading to an expression which can be analytically continued to all values of k. After some algebraic manipulations, we obtain the following corollary to Conjecture 4.1:

Conjecture 4.2. *For all k, the generating function as a formal power series in $\mathbb{Q}[k][[t]]$ is given by*

$$G(t,-k) = \frac{2^{2k-1}}{\sqrt{\pi}}\frac{\Gamma(k)\Gamma((1/2)+k)}{\Gamma(2k+1)}\left(\frac{\sin(t/2)}{t/2}\right)^{2k}\frac{1}{\sin(t/2)}$$

$$\times\left[\sin\left(\frac{t}{2}\right)+\Re(i\,\mathrm{e}^{it/2}{}_2F_1(1,-k,k,-\mathrm{e}^{-it}))\right],$$

(4.17)

where $\Re$ denotes the real part and $_2F_1(a,b;c;z)$ is the generalized hypergeometric function defined as

$$_2F_1(a,b;c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$. (For k non-positive integers and half-integers, the below expression of $G(t,-k)$ appears to be divergent. For these cases, one might try first expanding $G(t,-k)$ in t and setting k equal to the desired values.)

We have checked that our conjectural formula (4.7) indeed reproduces all the terms—see (4.10) for examples— generated by Faber's Maple program [4] up to $\mathcal{O}(t^{12})$.

4.3. Possible extensions

Motivated by our results, let us consider a similar generating function for the case of more marked points:

$$G_n(t,k) := \frac{n!}{(2n-2)!}H_{0,n} + \sum_{g\geq 1}t^{2g}\sum_{i=0}^{g}k^i\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-i}}{(1-\psi_1)\cdots(1-\psi_n)}.$$

At $k = -1$, it can be evaluated in terms of simple Hurwitz numbers as

$$G_n(t,-1) = n!\sum_{g=0}^{\infty}\frac{(-1)^g H_{g,n}}{(2g+2n-2)!}t^{2g}.$$

Interestingly, our previous generating function for simple Hurwitz numbers (3.2), with $\lambda = it$, is related to $G_n(t, -1)$:

$$\Phi(0)|_{\lambda=it} = \log Z(0) = \sum_{n \geq 1} \frac{(it)^{2n-2}}{n!} G_n(t, -1) q^n.$$

Hence, we have

$$G_n(t, -1) = \frac{n!}{t^{2n-2}} \sum_{k=1}^{n} \frac{(-1)^{k-n}}{n} \sum_{\substack{m_1 + \cdots + m_k = n \\ m_i > 0}} W_{m_i} \cdots W_{m_k},$$

where $W_1 = 1$ and

$$W_{m_i} = \sum_{\gamma \in \mathcal{R}_{m_i}} \left(\frac{f^\gamma}{m_i!} \right)^2 \cos \left[\binom{m_i}{2} \frac{\chi_\gamma(2)}{f^\gamma} t \right].$$

This relation might suggest a possible connection between the symmetric group S_n and the geometry of the moduli space of marked Riemann surfaces.

Of course, $G_n(t, -1)$ can be also explicitly computed from our previous computations of the simple Hurwitz numbers $H_{g,n}$. For example, we find that

$$G_3(t, -1) = \frac{(2 + \cos(t))}{3} \left(\frac{\sin(t/2)}{t/2} \right)^4,$$

$$G_4(t, -1) = \frac{(20 + 21 \cos(t) + 6 \cos(2t) + \cos(3t))}{12} \left(\frac{\sin(t/2)}{t/2} \right)^6,$$

etc. Similarly, $G_n(t, 0)$ can be computed by using the λ_g-conjecture. For example, one can easily show that

$$G_3(t, 0) = \frac{(3t/2)}{\sin(3t/2)},$$

etc. Although we are able to compute the generating function $G_n(t, k)$ at these particular values, it seems quite difficult—nevertheless possible—to determine its closed-form expression for all k. It would be a very intriguing project to search for the answer.

5. Conclusion, or an epilogue of questions unanswered

To recapitulate, the first part of our paper studies the simple branched covers of Riemann surfaces by Riemann surfaces of arbitrary genera. Upon fixing the degree of the irreducible covers, we have obtained closed form answers for simple Hurwitz numbers for arbitrary source and target Riemann surfaces, up to degree 7. For higher degrees, we have given a general prescription for extending our results. Our computations are

novel in the sense that the previously known formulas fix the genus of the source and target curves and vary the degree as a free parameter. Furthermore, by relating the simple Hurwitz numbers to descendant Gromov–Witten invariants, we have obtained the explicit generating functions (3.5) for the number of inequivalent reducible covers for arbitrary source and target Riemann surfaces. For an elliptic curve target, the generating function (3.3) is known to be a sum of quasi-modular forms. More precisely, in the expansion

$$Z = \sum_{n=0}^{\infty} A_n(q)\lambda^{2n},$$

the series $A_n(q)$ are known to be quasi-modular of weight $6n$ under the full modular group PSL$(2, \mathbb{Z})$. Our general answer (3.5) for an arbitrary target genus differs from the elliptic curve case only by the prefactor $(n!/f^\gamma)^{2h-2}$. Naively, it is thus tempting to hope that the modular property persists, so that in the expansion

$$Z(h) = \sum_{n=0}^{\infty} A_n^h(q)\lambda^{2n},$$

the series $A_n^h(q)$ are quasi-automorphic forms, perhaps under a genus-h subgroup of PSL$(2, \mathbb{Z})$.

Throughout the paper, we have taken caution to distinguish two different conventions of accounting for the automorphism groups of the branched covers and have clarified their relations when possible. The recent developments in the study of Hurwitz numbers involve connections to the relative Gromov–Witten theory and Hodge integrals on the moduli space of stable curves. In particular, Li et al. have obtained a set of recursion relations for the numbers $\mu_{h,w}^{g,n}(\alpha)$ by applying the gluing formula to the relevant relative GW invariants [13]. Incidentally, these recursion relations require as initial data the knowledge of simple Hurwitz numbers, and our work would be useful for applying the relations as well.

Although we cannot make any precise statements at this stage, our work may also be relevant to understanding the conjectured Toda hierarchy and the Virasoro constraints for Gromov–Witten invariants on $\mathbb{P}^1$ and elliptic curve. It has been shown in [18] that Virasoro constraints lead to certain recursion relations among simple Hurwitz numbers for a $\mathbb{P}^1$ target. It might be interesting to see whether there exist further connections parallel to these examples. The case of an elliptic curve target seems, however, more elusive at the moment. The computations of the Gromov–Witten invariants for an elliptic curve are much akin to those occurring for Calabi–Yau three-folds. For instance, a given n-point function receives contributions from the stable maps of all degrees, in contrast to the Fano cases in which only a finite number of degrees yields the correct dimension of the moduli space. Consequently, the recursion relations and the Virasoro constraints seem to lose their efficacy when one considers the Gromov–Witten invariants of an elliptic curve. It is similar to the ineffectiveness of the WDVV equations for determining the number of rational curves on a Calabi–Yau three-fold.

Appendix A. Rudiments of the symmetric group S_n

It is well known that the conjugacy classes, and thus the irreducible representations, of the symmetric group S_n are in one-to-one correspondence with distinct ordered partitions of n. Let us consider an irreducible representation of S_n labeled by the ordered partition $\gamma = (n_1, \ldots, n_m) \vdash n$, where $n_1 \geq n_2 \geq \cdots \geq n_m$. Let $p_i = n_i + m - i$ and define the Van der Monde determinant

$$
D(p_1, \ldots, p_m) \equiv
\begin{vmatrix}
p_1^{m-1} & p_1^{m-2} & \cdots & p_1 & 1 \\
p_2^{m-1} & p_2^{m-2} & \cdots & p_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
p_m^{m-1} & p_m^{m-2} & \cdots & p_m & 1
\end{vmatrix}.
$$

Then, the irreducible characters evaluated at the conjugacy classes (1^n) and (2) can be written, respectively, as

$$
\chi_\gamma(1^n) = \frac{n!}{p_1! p_2! \cdots p_m!} D(p_1, \ldots, p_m),
$$

and

$$
\chi_\gamma(2) = (n-2)! \sum_{i \in \mathcal{I}} \frac{D(p_1, \ldots, p_{i-1}, p_i - 2, p_{i+1}, \ldots, p_m)}{p_1! \cdots p_{i-1}!(p_i - 2)! p_{i+1}! \cdots p_m!},
$$

where the index set $\mathcal{I}$ is defined as $\{i \in \{1, \ldots, m\} | (p_i - 2) \geq 0\}$. Furthermore, these irreducible characters satisfy the simple relation

$$
\binom{n}{2} \frac{\chi_\gamma(2)}{\chi_\gamma(1^n)} = \frac{1}{2} \sum_{k=1}^{m} n_k(n_k + 1) - \sum_{k=1}^{m} k \cdot n_k, \tag{A.1}
$$

which we utilize in the paper.

Appendix B. Useful facts

Lemma B.1. *Let $L = \langle e_1, e_2 \rangle := \mathbb{Z}e_1 + \mathbb{Z}e_2$ be a two-dimensional lattice generated by e_1 and e_2. Then, the number of inequivalent sublattices $L' \subset L$ of index $[L : L'] = n$ is given by $\sigma_1(n) := \sum_{d|n} d$.*

Proof. Let $f_1 = de_1 \in L'$ be the smallest multiple of e_1. Then, there exists $f_2 = ae_1 + be_2 \in L'$, $a < d$, such that L' is generated by f_1 and f_2 over $\mathbb{Z}$. It is clear that the index of this lattice is db. Thus, for each d dividing the index n, we have the following d inequivalent sublattices: $\langle de_1, (n/d)e_2 \rangle$, $\langle de_1, e_1 + (n/d)e_2 \rangle$, ..., $\langle de_1, (d-1)e_1 + (n/d)e_2 \rangle$. The lemma now follows. $\square$

Lemma B.2. *Let $\pi(m)$ be the number of distinct ordered partitions of a positive integer m into positive integers. Then, the function $\sigma_1(n)$ has the following expression:*

$$\sigma_1(n) = n \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{m_1 + \cdots + m_k = n} \left(\prod_{i=1}^{k} \pi(m_i) \right).$$

Proof. As is well known, the functions $\pi(m)$ arise as coefficients of the expansion of $q^{1/24} \eta(q)^{-1}$, i.e.

$$\frac{q^{1/24}}{\eta(q)} = 1 + \sum_{m=1}^{\infty} \pi(m) q^m. \tag{B.1}$$

We take log of both sides of (B.1) and q-expand the resulting expression on the right hand side. Now, using the fact that

$$\log(q^{1/24} \eta(q)^{-1}) = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n,$$

we match the coefficients of q^n to get the desired result. $\square$

As in [17], let $N(n, m, \nu)$ be the number of edge-ordered graphs with n vertices, m edges, and ν cycle partition, and $N_c(n, m, \nu)$ the number of connected such graphs. Then,

Lemma B.3. $N(2k, 2\ell, (k, k)) = 0$ *for* $\ell \leq k - 2$.

Proof. These constraints follow from Theorem 4 of [17] which states that the length l of the cycle partition must satisfy the conditions $c \leq l \leq \min(n, m - n + 2c)$ and $l = m - n \pmod{2}$, where c is the number of connected components. In our case, $l = 2$ and the second condition is always satisfied. The first condition, however, is violated for all $\ell \leq k - 2$ because $c \leq 2$ and thus $\min(2\ell - 2k + 2c) \leq 0$. $\square$

Similarly, one has the following lemma.

Lemma B.4. $N_c(2k, 2k - 2, (k, k)) = 0$.

Proof. This fact again follows from Theorem 4 of [17]. Here, $c = 1$ and $\min(n, m - n + 2c) = 0$, whereas $l = 2$, thus violating the first condition of the theorem. $\square$

Appendix C. Computation of simple Hurwitz numbers

For computations of $\tilde{\mu}_{h,n}^{g,n} = N_{n,h,r}$, we will need the following relation among the numbers of irreducible and reducible covers [14]:

$$T_{n,h,\sigma} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1 + \cdots + n_k = n \\ \sigma_1 + \cdots + \sigma_k = \sigma}} \binom{n}{n_1, \ldots, n_k} B_{n_1,h,\sigma_1} \cdots B_{n_k,h,\sigma_k}, \qquad \text{(C.1)}$$

where

$$B_{n,h,r} = (n!)^{2h-1} \binom{n}{2}^{r} \left[\sum_{\gamma \in \mathcal{R}_n} \frac{1}{(f^\gamma)^{2h-2}} \left(\frac{\chi_\gamma(2)}{f^\gamma} \right)^{r} \right].$$

Furthermore, in these computations, we assume that r is positive unless indicated otherwise.

C.1. Degrees 1 and 2

It is clear that the degree-1 simple Hurwitz numbers are given by

$$\mu_{h,1}^{g,1}(1) = \delta_{g,h}.$$

The number of simple double covers of a genus-h Riemann surface by genus-g Riemann surfaces can be obtained by using the work of Mednykh on Hurwitz numbers for the case where all branchings have the order equal to the degree of the covering [15].

Claim C.1. *The simple Hurwitz numbers $\tilde{\mu}_{h,2}^{g,2}(1, 1)$ are equal to 2^{2h} for $g > 2(h - 1) + 1$.*

Proof. For $g > 2(h - 1) + 1$, the number r of simple branch points is positive, and we can use the results of Mednykh [14]. Let p be a prime number and D_p the set of all irreducible representations of the symmetric group S_p. Then, Mednykh shows that the number $N_{p,h,r}$ of inequivalent degree-p covers of a genus-h Riemann surface by genus-g Riemann with r branch points (The Riemann–Hurwitz formula determines r to be $r = [2(1-h)p+2(g-1)]$.) of order-p is given by

$$N_{p,h,r} = \frac{1}{p!} T_{p,h,r} + p^{2h-2}[(p - 1)^r + (p - 1)(-1)^r],$$

where

$$T_{p,h,r} = p! \sum_{\gamma \in D_p} \left(\frac{\chi_\gamma(p)}{p} \right)^{r} \left(\frac{p!}{f^\gamma} \right)^{2h-2+r},$$

where $\chi_\gamma(p)$ is the character of a p-cycle in the irreducible representation γ of S_p and f^γ is the dimension of γ. For $p = 2$, S_2 is isomorphic to $\mathbb{Z}_2$, and the characters of the transposition for two one-dimensional irreducible representations are 1 and -1, respectively. It follows that

$$N_{2,h,r} = T_{2,h,r} = \begin{cases} 2^{2h} & \text{for } r \text{ even,} \\ 0 & \text{for } r \text{ odd,} \end{cases}$$

and therefore that

$$\tilde{\mu}_{h,2}^{g,2}(1,1) = N_{2,h,r} = 2^{2h},$$

which is the desired result. $\qquad\square$

Remark. The answer for the case $g = 1$ and $h = 1$ is 3, which follows from Lemma B.1. For $h = 1$ and $g > 1$, we have $\tilde{\mu}_{1,2}^{g,2}(1,1) = 4$.

C.2. Degree 3

The following lemma will be useful in the ensuing computations:

Lemma C.1. Let $t_k^p = 2\delta_{k,1} \sum_{i=1}^j \delta_{p,i} + \delta_{k,2} \sum_{i=j+1}^r \delta_{p,i}$. Then,

$$B_{2,h,(t_k^p)} = \begin{cases} 2^{2h} & \text{for } j \text{ even,} \\ 0 & \text{for } j \text{ odd.} \end{cases}$$

Proof. The result follows trivially from the general formula for $B_{n,h,\sigma}$ by noting that the character values of the transposition for the two irreducible representations of S_2 are 1 and -1. $\qquad\square$

We now show the following claim.

Claim C.2. The degree-3 simple Hurwitz numbers are given by

$$N_{3,h,r} = 2^{2h-1}(3^{2h-2+r} - 1) = 2^{2h-1}(3^{2g-4h+2} - 1),$$

where $r = 6(1 - h) + 2(g - 1)$ is the number of simple branch points.

Proof. $T_{3,h,r}$ receives non-zero contributions from the following partitions of 3: (3) and (1, 2). There are three irreducible representations of S_3 of dimensions 1, 1, and 2, whose respective values of their characters on a transposition are 1, -1, and 0. Taking care to account for the correct combinatorial factors easily yields the desired result. $\qquad\square$

C.3. Degree 4

The degree 4 answer is slightly more complicated.

Claim C.3. *The degree-4 simple Hurwitz numbers are given by*

$$
\begin{aligned}
N_{4,h,r} &= 2^{2h-1}[(3^{2h-2+r}+1)2^{4h-4+r} - 3^{2h-2+r} - 2^{2h-3+r}+1] + 2^{4h-4+r}(2^{2h}-1) \\
&= 2^{2h-1}[(3^{2g-6h+4}+1)2^{2g-4h+2} - 3^{2g-6h+4} - 2^{2g-6h+3}+1] \\
&\quad + 2^{2g-4h+2}(2^{2h}-1).
\end{aligned} \tag{C.2}
$$

Proof. The last term in (C.2) comes from the second term in (2.6) by applying Lemma 2.1. To compute $T_{4,h,r}$, we note that the only consistent partitions of 4 and σ are when 4 has the following partitions: (4), $(1,3)$, $(2,2)$, and $(1,1,2)$. The only non-immediate sum involves

$$
\sum_{\sigma_1+\sigma_2=\sigma} B_{2,h,\sigma_1} B_{2,h,\sigma_2},
$$

which, upon applying Lemma C.1, becomes 2^{4h+r-1}. □

Higher degree computations are similarly executed, although one must keep track of some combinatorial factors arising from inequivalent distributions of σ in (C.1), and we thus omit their proofs in the subsequent discussions.

C.4. Degree 5

For the degree 5 computation, we need the following lemma.

Lemma C.2. *Let $t_k^p = 3\delta_{k,1}\sum_{i=1}^{j}\delta_{i,p} + (\delta_{k,1}+\delta_{k,2})\sum_{i=j+1}^{r}$. Then,*

$$
B_{3,h,(t_k^p)} = \begin{cases}
2^{2h} \times 3^{2h-1+r-j} & \text{for } j < r \text{ even,} \\
0 & \text{for } j \text{ odd,} \\
2 \times 3^{2h-1}(2^{2h-1}+1) & \text{for } j = r,
\end{cases}
$$

from which follows the following claim.

Claim C.4. *The degree 5 simple Hurwitz numbers are given by*

$$
\begin{aligned}
N_{5,h,r} &= 2^{2h-1}(2^{2h+r-2} - 2^{4h+r-4} - 1) - 3^{2h-2} \times 2^{2h-1}(1 + 2^{2h+r-2} + 2^{2h+2r-2}) \\
&\quad + 3^{2h+r-2} \times 2^{2h-1}(1 - 2^{4h+r-4}) + 2^{6h+r-5} \times 3^{2h-2} \\
&\quad + (1 + 2^{4h+r-4})2^{2h-1} \times 3^{2h-2} \times 5^{2h+r-2}.
\end{aligned}
$$

C.5. Degree 6

Similarly, by using the following lemma

Lemma C.3. *Let* $t_k^p = 4\delta_{k,1} \sum_{i=1}^{j} \delta_{i,p} + (\delta_{k,1} + \delta_{k,2}) \sum_{i=j+1}^{r}$. *Then,*

$$
B_{4,h,(t_k^p)} = \begin{cases} 3 \times 2^{r-j+6h-2}(3^{2h-2+r-j} + 1) & \text{for } j < r \text{ even}, \\ 0 & \text{for } j \text{ odd}, \\ 3 \times 2^{4h-1}(2^{2h-1}3^{2h-2} + 2^{2h-1} + 3^{2h-2}) & \text{for } j = r, \end{cases}
$$

we obtain the following claim.

Claim C.5. *The degree-6 simple Hurwitz numbers are given by*

$$
\begin{aligned}
N_{6,h,r} = \frac{1}{720} \Bigg[&360 \times 2^{2h} - 135 \times 2^{4h+r} - 40 \times 2^{2h} \times 3^{2h+r} \\
&- \frac{5 \times 2^{2h} \times 3^{4h+r}(8 + 2^{2h+r})}{9} + 20 \times 2^{2h} \times 3^{2h}(4 + 2^{2(h+r)} + 2^{2h+r}) \\
&+ \frac{15 \times 2^{6h}(3 + 3^r)}{2} + \frac{5 \times 2^{6h+r}(9 + 3^{2h+r})}{2} \\
&- \frac{2^{2h} \times 3^{2h}(25 \times 2^{4h+r} + 16 \times 5^{2h+r} + 2^{4h+r} \times 5^{2h+r})}{10} \\
&+ \frac{\begin{aligned}2^{6h}(100 \times 3^{4h+r} + 25 \times 2^{2h} \times 3^{4h+r} + 25 \times 2^{2h} \times 3^{4h+2r} \\ + 81 \times 2^{2h} \times 5^{2h+r} + 2^{2h} \times 3^{4h+r} \times 5^{2h+r})\end{aligned}}{360} \\
&- \frac{\begin{aligned}5 \times 2^{6h}(9 \times 2^{2h} + 4 \times 3^{2h} + 9 \times 2^{2h} \times 3^r \\ + 2^{2h} \times 3^{2h} \times 5^r + 2^{2h} \times 3^{2h} \times 7^r)\end{aligned}}{8} \Bigg] \\
&+ 2^{6h-5} \times 3^{r-1}[3^{2h-1}(2^{2h-1} + 1) - 3(2^{2h-1}) + 1].
\end{aligned}
$$

C.6. Degree 7

Claim C.6. *The degree 7 simple Hurwitz numbers are given by*

$$
\begin{aligned}
N_{7,h,r} = &\frac{-2^{2h}}{2} - \frac{3 \times 2^{6h}}{32} + \frac{2^{8h}}{64} + \frac{2^{4h+r}}{4} - \frac{2^{6h+r}}{32} - \frac{2^{2h} \times 3^{2h}}{6} + \frac{2^{6h} \times 3^{2h}}{96} \\
&- \frac{2^{8h} \times 3^{2h}}{576} - \frac{2^{4h+r} \times 3^{2h}}{36} - \frac{2^{4h+2r} \times 3^{2h}}{24} + \frac{2^{6h} \times 5^r \times 3^{2h}}{288} \\
&- \frac{2^{8h} \times 5^r \times 3^{2h}}{1152} - \frac{2^{6h} \times 3^r}{32} + \frac{2^{8h} \times 3^r}{64} + \frac{2^{2h} \times 3^{2h+r}}{18} \\
&+ \frac{2^{6h} \times 3^{2h+r}}{144} - \frac{2^{8h} \times 3^{2h+r}}{1152} - \frac{2^{6h+r} \times 3^{2h+r}}{288} + \frac{2^{2h} \times 3^{4h+r}}{81}
\end{aligned}
$$

$$- \frac{2^{6h} \times 3^{4h+r}}{1296} - \frac{2^{8h} \times 3^{4h+r}}{10\,368} + \frac{2^{4h+r} \times 3^{4h+r}}{648} - \frac{2^{6h+r} \times 3^{4h+r}}{2592}$$

$$- \frac{2^{8h} \times 3^{4h+2r}}{5184} - \frac{2^{6h} \times 5^{2h}}{800} + \frac{2^{8h} \times 3^{2h} \times 5^{2h}}{28\,800} - \frac{2^{4h+2r} \times 3^{2h} \times 5^{2h}}{1800}$$

$$- \frac{2^{8h} \times 3^{2(h+r)} \times 5^{2h}}{28\,800} - \frac{2^{4h+r} \times 3^{2h+r} \times 5^{2h}}{1800} + \frac{2^{6h+r} \times 3^{4h+r} \times 5^{2h}}{64\,800}$$

$$+ \frac{2^{6h} \times 3^{4h+2r} \times 5^{2h}}{64\,800} + \frac{2^{8h} \times 3^{2h} \times 5^{r}}{576} - \frac{2^{8h} \times 5^{2h+r}}{3200}$$

$$+ \frac{2^{2h} \times 3^{2h} \times 5^{2h+r}}{450} + \frac{2^{6h+r} \times 3^{2h} \times 5^{2h+r}}{7200} - \frac{2^{8h} \times 3^{4h+r} \times 5^{2h+r}}{259\,200}$$

$$+ \frac{2^{8h} \times 3^{2h} \times 7^{r}}{576} + \frac{2^{8h} \times 3^{2h} \times 7^{2h+r}}{56\,448} + \frac{2^{6h+r} \times 3^{2h} \times 5^{2h} \times 7^{2h+r}}{352\,800}$$

$$+ \frac{2^{8h} \times 3^{4h+r} \times 5^{2h} \times 7^{2h+r}}{12\,700\,800} - \frac{2^{8h} \times 3^{2h} \times 5^{2h} \times 11^{r}}{28\,800}.$$

Appendix D. Reducible covers

$$B_{n,h,r} = (n!)^{2h-1} \binom{n}{2}^{r} \left[\sum_{\gamma \in \mathcal{R}_n} \frac{1}{(f^{\gamma})^{2h-2}} \left(\frac{\chi_{\gamma}(2)}{f^{\gamma}} \right)^{r} \right],$$

$$B_{2,h,r} = 2 \times 2^{2h-1},$$

$$B_{3,h,r} = 2 \times 3!^{2h-1} \binom{3}{2}^{r},$$

$$B_{4,h,r} = 2 \times 4!^{2h-1} \binom{4}{2}^{r} \left[1 + \frac{1}{3^{2h-2+r}} \right],$$

$$B_{5,h,r} = 2 \times 5!^{2h-1} \binom{5}{2}^{r} \left[1 + \frac{2^{r}}{4^{2h-2+r}} + \frac{1}{5^{2h-2+r}} \right],$$

$$B_{6,h,r} = 2 \times 6!^{2h-1} \binom{6}{2}^{r} \left[1 + \frac{3^{r}}{5^{2h-2+r}} + \frac{3^{r}}{9^{2h-2+r}} + \frac{2^{r}}{10^{2h-2+r}} + \frac{1}{5^{2h-2+r}} \right],$$

$$B_{7,h,r} = 2 \times 7!^{2h-1} \binom{7}{2}^{r} \left[1 + \frac{4^{r}}{6^{2h-2+r}} + \frac{6^{r}}{14^{2h-2+r}} + \frac{5^{r}}{15^{2h-2+r}} + \frac{4^{r}}{14^{2h-2+r}} \right.$$

$$\left. + \frac{5^{r}}{35^{2h-2+r}} + \frac{1}{21^{2h-2+r}} \right],$$

$$B_{8,h,r} = 2 \times 8!^{2h-1} \binom{8}{2}^r \left[1 + \frac{5^r}{7^{2h-2+r}} + \frac{10^r}{20^{2h-2+r}} + \frac{9^r}{21^{2h-2+r}} + \frac{10^r}{28^{2h-2+r}} \right.$$
$$\left. + \frac{16^r}{64^{2h-2+r}} + \frac{5^r}{35^{2h-2+r}} + \frac{4^r}{14^{2h-2+r}} + \frac{10^r}{70^{2h-2+r}} + \frac{4^r}{56^{2h-2+r}} \right],$$

$$B_{9,h,r} = 2 \times 9!^{2h-1} \binom{9}{2}^r \left[1 + \frac{6^r}{8^{2h-2+r}} + \frac{15^r}{27^{2h-2+r}} + \frac{14^r}{28^{2h-2+r}} + \frac{20^r}{48^{2h-2+r}} \right.$$
$$+ \frac{35^r}{105^{2h-2+r}} + \frac{14^r}{56^{2h-2+r}} + \frac{14^r}{42^{2h-2+r}} + \frac{36^r}{162^{2h-2+r}} + \frac{20^r}{120^{2h-2+r}}$$
$$\left. + \frac{21^r}{189^{2h-2+r}} + \frac{14^r}{84^{2h-2+r}} + \frac{14^r}{168^{2h-2+r}} + \frac{6^r}{216^{2h-2+r}} \right].$$

Appendix E. Simple Hurwitz numbers for an elliptic curve target

We can compare our answers in the case of an elliptic curve target with those obtained from string theory. To do so, we organize $T_{n,1,2g-2}/n!$ into a generating function $H_1^g(q)$, which is defined as

$$(2g-2)! H_1^g = \sum_{n=1}^{\infty} \mu_{1,n}^{g,n} q^n = \sum_{n=1}^{\infty} \frac{T_{n,1,2g-2}}{n!} q^n.$$

Our explicit formulas for $T_{n,1,2g-2}/n!$, $n \le 7$, from Section C and the recursive method discussed in Section 2.4 give rise to the following q-expansions of $H_1^g(q)$:

$$2! H_1^2 = 2q^2 + 16q^3 + 60q^4 + 160q^5 + 360q^6 + 672q^7 + 1240q^8$$
$$+ 1920q^9 + \mathcal{O}(q^{10}),$$

$$4! H_1^3 = 2q^2 + 160q^3 + 2448q^4 + 18\,304q^5 + 90\,552q^6 + 341\,568q^7 + 1\,068\,928q^8$$
$$+ 2\,877\,696q^9 + \mathcal{O}(q^{10}),$$

$$6! H_1^4 = 2q^2 + 1456q^3 + 91\,920q^4 + 1\,931\,200q^5 + 21\,639\,720q^6 + 160\,786\,272q^7$$
$$+ 893\,985\,280q^8 + 4\,001\,984\,640q^9 + \mathcal{O}(q^{10}),$$

$$8! H_1^5 = 2q^2 + 13\,120q^3 + 3\,346\,368q^4 + 197\,304\,064q^5 + 5\,001\,497\,112q^6$$
$$+ 73\,102\,904\,448q^7 + 724\,280\,109\,568q^8 + 5\,371\,101\,006\,336q^9 + \mathcal{O}(q^{10}),$$

$$10! H_1^6 = 2q^2 + 118\,096q^3 + 120\,815\,280q^4 + 19\,896\,619\,840q^5$$
$$+ 1\,139\,754\,451\,080q^6 + 32\,740\,753\,325\,472q^7 + 577\,763\,760\,958\,720q^8$$
$$+ 7\,092\,667\,383\,039\,360q^9 + \mathcal{O}(q^{10}),$$

$$12!H_1^7 = 2q^2 + 1\,062\,880q^3 + 4\,352\,505\,888q^4 + 1\,996\,102\,225\,024q^5$$
$$+ 258\,031\,607\,185\,272q^6 + 14\,560\,223\,135\,464\,128q^7$$
$$+ 457\,472\,951\,327\,051\,008q^8 + 9\,293\,626\,316\,677\,061\,376q^9 + \mathcal{O}(q^{10}),$$

$$14!H_1^8 = 2q^2 + 9\,565\,936q^3 + 156\,718\,778\,640q^4 + 199\,854\,951\,398\,080q^5$$
$$+ 58\,230\,316\,414\,059\,240q^6 + 6\,451\,030\,954\,702\,152\,672q^7$$
$$+ 360\,793\,945\,093\,731\,688\,960q^8 + 12\,127\,449\,147\,074\,861\,834\,880q^9$$
$$+ \mathcal{O}(q^{10}),$$

$$16!H_1^9 = 2q^2 + 86\,093\,440q^3 + 5\,642\,133\,787\,008q^4 + 19\,994\,654\,452\,125\,184q^5$$
$$+ 13\,120\,458\,818\,999\,011\,032q^6 + 2\,852\,277\,353\,239\,208\,548\,608q^7$$
$$+ 283\,889\,181\,859\,169\,785\,013\,248q^8$$
$$+ 15\,786\,934\,495\,235\,533\,394\,850\,816q^9 + \mathcal{O}(q^{10}),$$

$$18!H_1^{10} = 2q^2 + 774\,840\,976q^3 + 203\,119\,138\,758\,000q^4$$
$$+ 1\,999\,804\,372\,817\,081\,920q^5 + 2\,954\,080\,786\,719\,122\,704\,200q^6$$
$$+ 1\,259\,649\,848\,110\,685\,616\,355\,872q^7$$
$$+ 223\,062\,465\,532\,295\,875\,789\,024\,000q^8$$
$$+ 20\,519\,169\,517\,386\,068\,841\,434\,851\,200q^9 + \mathcal{O}(q^{10}),$$

The free energies $\mathcal{F}_g$ of string theory on the target space of an elliptic curve are known to be quasi-modular forms of weight $6g - 6$. They have been computed up to genus 8 in [16] and have the same expansions in $q = \exp(\hat{t})$, where $\hat{t}$ is the Kähler parameter of the elliptic curve, as what we have above for H_1^g.

For convenience, we also summarize the simple Hurwitz numbers for an elliptic curve target and arbitrary source Riemann surfaces up to degree 7:

$$\mu_{1,1}^{g,1}(1) = \delta_{g,1}, \qquad \mu_{1,2}^{g,2}(1^2) = 2, \qquad \mu_{1,3}^{g,3}(1^3) = 2[3^r - 1],$$
$$\mu_{1,4}^{g,4}(1^4) = 2[6^r + 2^{r-1} - 3^r + 1], \qquad \mu_{1,5}^{g,5}(1^5) = 2[10^r - 6^r + 5^r - 4^r + 3^r - 2],$$
$$\mu_{1,6}^{g,6}(1^6) = 2 \times 15^r - 2 \times 10^r + 2 \times 9^r - 2 \times 7^r + 6^r - 2 \times 5^r + 4 \times 4^r$$
$$- 4 \times 3^{r-1} + 2^r + 4,$$
$$\mu_{1,7}^{g,7}(1^7) = 2[21^r - 15^r + 14^r - 11^r + 10^r - 2 \times 9^r + 3 \times 7^r - 6^r + 2 \times 5^r$$
$$- 4 \times 4^r + 2 \times 3^r - 2^r - 4], \tag{E.1}$$

where $r = 2g - 2$.

Appendix F. The Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$

$$\mu_{0,2}^{1,2k}(k,k) = \frac{(2k+2)!}{48(k!)^2}(3k-2)k^{2k+1},$$

$$\mu_{0,2}^{2,2k}(k,k) = \frac{(2k+4)!}{2(k!)^2}\left(\frac{28-73k+49k^2}{2880}\right)k^{2k+3},$$

$$\mu_{0,2}^{3,2k}(k,k) = \frac{(2k+6)!}{2(k!)^2}\left(\frac{-744+2530k-2949k^2+1181k^3}{725\,760}\right)k^{2k+5},$$

$$\mu_{0,2}^{4,2k}(k,k) = \frac{(2k+8)!}{2(k!)^2}\left(\frac{18\,288-72\,826k+111\,309k^2-77\,738k^3+21\,015k^4}{174\,182\,400}\right)$$
$$\times k^{2k+7}.$$

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